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Summary of the Ph.D. thesis entitled

Arborescence packing and restricted b-matchings

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Introduction

The thesis has two main topics, the first of them is arborescence packing. We consider extensions of Edmonds' fundamental result on packing disjoint spanning arborescences. The problem can be naturally generalized in two directions: the edge-disjointness condition may be strengthened, and the set of nodes spanned by the arborescences may be decreased.

- *We give a disproof of the conjecture of Colussi, Conforti and Zambelli on strongly edge-disjoint arborescences.* For $k = 2$ the conjecture is true; we give its generalization for dicycle-disjoint Steiner arborescences.

- *We present a linear time algorithm for finding a pair of disjoint in- and out-arborescences in an acyclic digraph.* Deciding the existence of such arborescences is NP-complete in general. Our algorithm is based on a reduction to bipartite matching in an associated bipartite graph.

- *We present a strongly polynomial time algorithm for finding disjoint arborescences spanning convex sets under capacity constraints.* Our solution is based on the deep understanding of the connection between packing arborescences and covering intersecting bi-set families.

- *We give a polyhedral description of arborescence packable subgraphs and prove that the system is TDI.* The proof strongly relies on the special intersecting bi-set families appearing in the proof of Fujishige's theorem.

The second part of the thesis deals with restricted b-matchings, mainly with C_k-free k-matchings. It has been known that the C_k-free 2-matching problem is NP-complete for $k \geq 5$. We consider the C_3-free and the C_4-free 2-matching, and the $K_{t,t}$- and K_{t+1}-free t-matching problems in graphs that satisfy certain degree bounds.

- *We give a min-max theorem and an algorithm for the square-free 2-matching problem in subcubic graphs.* We show that the weighted version of the problem is NP-hard even in planar bipartite cubic graphs, but is polynomially solvable when the weight function is node-induced on each square.

- *We give a min-max theorem and an algorithm for the $K_{t,t}$- and K_{t+1}-free t-matching problem in degree bounded graphs.* Note that this problem is a generalization of the C_3-free, C_4-free and $C_{\leq 4}$-free 2-matching problems.

- *We give a description of the triangle-free 2-matching polytope of subcubic graphs.* The description was conjectured by Hartvigsen and Li; the complete proof appeared recently. We give an independent proof of the result which relies on a shrinking method.

The last chapter examines arbitrary triangle-free subgraphs, that is, when the degree bound on the nodes in the subgraph is omitted. The problem is approached through shadow systems and Turán numbers.

- *We prove that the set of multisets with size k over a ground set with size also k has the so-called splitting property.* From this, we show that a weighted extension of the Turán number admits the same upper bounds as the unweighted one. We also prove a combinatorial colouring theorem and a fractional version of an extension of Tuza's conjecture to hypergraphs.

Packing arborescences

In a directed graph $D = (V, A)$ an r**-arborescence** is a directed tree in which every node is reachable from a given root node r. We call D **rooted** k**-edge-connected** if for each $v \in V$, there exist k edge-disjoint directed paths from r to v. Edmonds' fundamental theorem characterizes the existence of k edge-disjoint spanning arborescences rooted at the same node [14].

Theorem 1 (Edmonds). *Let $D = (V, A)$ be a digraph with root r. D has k edge-disjoint spanning r-arborescences if and only if D is rooted k-edge-connected.*

One way to extend Edmonds' theorems is to decrease the size of the node sets spanned by the arborescences in question. However, it is not easy to find such a generalization as one can easily run into difficult questions. We proved the followings.

Theorem 2 (B. and Frank [3]). *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R. The problem of deciding whether there are k disjoint arborescences so that they are rooted at distinct nodes in R and each of them spans T is NP-complete.*

Theorem 3 (B. and Frank [3]). *Let $D = (V, A)$ be a digraph with $u_1, u_2, v_1, v_2 \in V$ and let $U_1 = V$, $U_2 = V - v_1$. The problem of finding two edge-disjoint arborescences rooted at u_1, u_2 and spanning U_1, U_2, respectively, is NP-complete.*

However, in 2009, Kamiyama, Katoh and Takizawa [18] gave a surprising new proper extension of Edmonds' theorem which has the following equivalent form, proved by Fujishige. For two disjoint subsets X and Y of V, we say that Y is **reachable** from X if there is a directed path in D whose first node is in X and last node is in Y. We call a subset U of nodes **convex** if there is no node v in $V \setminus U$ so that U is reachable from v and v is reachable from U.

Theorem 4 (Kamiyama, Katoh and Takizawa, Fujishige). *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, \dots, r_k\} \subseteq V$ be a list of k (possibly not distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. There are edge-disjoint r_i-arborescences F_i spanning U_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{1}$$

where $p_1(Z)$ denotes the number of sets U_i's for which $U_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

For an r-arborescence F, a node u is an F**-ancestor** of another node v if there is a directed path from u to v in F. We denote this unique path by $F(u, v)$. We say that a node w **dominates** a node v if every path from r to v includes w. Two arcs are called **symmetric** if they share the same endnodes but have opposite orientations. Two edge-disjoint arborescences F_1, F_2 rooted at r are called **strongly edge-disjoint** if the paths $F_1(r, v), F_2(r, v)$ do not contain a pair of symmetric arcs for each $v \in V$.

Conjecture 5 (Colussi, Conforti and Zambelli). *Let $D = (V, A)$ be a digraph with root r. D has k strongly edge-disjoint spanning r-arborescences if and only if D is rooted k-edge-connected.*

For $k = 2$, the conjecture was verified in [11]. We give a disproof of the conjecture for $k \geq 3$, and generalize the case $k = 2$ to dicycle-disjoint Steiner arborescences. For a terminal set $T \subseteq V$, an r-arborescence spanning T is called a **Steiner-arborescence**. Two Steiner-arborescences F_1 and F_2 are called **edge-independent** if the paths $F_1(r,t), F_2(r,t)$ are edge-disjoint for every terminal $t \in T$. We call two edge-independent Steiner-arborescences F_1 and F_2 **dicycle-disjoint** if for each $t \in T$ the union $F_1(r,t) \cup F_2(r,t)$ does not contain a directed cycle.

Theorem 6 (B. and Kovács [6]). *Let $D = (V, A)$ be a directed graph with root r and terminal set T. There exist two dicycle-disjoint Steiner-arborescences if and only if there exist two edge-disjoint paths from r to t for each $t \in T$.*

From now on, an r-out-arborescence is just the same as an r-arborescence defined earlier, while an r-in-arborescence is a directed tree in which the edges are directed toward the root node r. It is known [9] that the problem of finding a pair of edge-disjoint spanning r_1-in-arborescence and r_2-out-arborescence for given roots $r_1, r_2 \in V$ is NP-complete. We consider this problem in a directed acyclic graph and give a linear time algorithm for solving it.

Theorem 7 (B., Fujishige and Kamiyama [4]). *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, we can discern the existence of a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence, and find such arborescences if they exist, in $O(|A|)$ time.*

Covering intersecting bi-set families

Given a ground-set V, we call a pair $X = (X_O, X_I)$ of subsets a **bi-set** if $X_I \subseteq X_O \subseteq V$ where X_O is the **outer member** and X_I is the **inner member** of X. The **intersection** $\cap$ and the **union** $\cup$ of bi-sets is defined as follows: for bi-sets X, Y let $X \cap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \cup Y := (X_O \cup Y_O, X_I \cup Y_I)$. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$. A family of bi-sets is called **intersecting** if both the union and the intersection of any two intersecting members of $\mathcal{F}$ belong to $\mathcal{F}$. A directed edge **enters** or **covers** X if its head is in X_I and its tail is outside X_O. An edge set **covers** a family of bi-sets if it covers each member of the family.

There is another line of extending Theorem 1 in which, rather than working directly with arborescences, one considers disjoint edge-coverings of certain families of sets or bi-sets. The bi-set families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ said to satisfy the **mixed intersection** property if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

For a bi-set X, let $p_2(X)$ denote the number of indices i for which $\mathcal{F}_i$ contains X. In [10], we derived an extension of a theorem of Szegő on covering intersecting families to bi-set systems.

Theorem 8 (B. and Frank). *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting families of bi-sets on ground set V satisfying the mixed intersection property. The edges of D can be partitioned into k subsets $A_1, \ldots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p_2(X) \text{ for every bi-set } X.$$

We proved that Theorem 4 can be derived from Theorem 8. The application of bi-sets gives a new insight into the structure of convex sets. By using the special bi-set families appearing in the proof, we give a strongly polynomial time algorithm for finding rooted branchings spanning given convex sets under edge capacity constraints.

Theorem 9 (B. and Frank [3]). *Let $D = (V, A)$ be a digraph, $g : A \to \mathbb{Z}_+$ a capacity function, $\mathcal{R} = \{R_1, \ldots, R_k\}$ a list of root-sets, $\mathcal{U} = \{U_1, \ldots, U_k\}$ a set of convex sets with $R_i \subseteq U_i$ and $m : \mathcal{R} \to \mathbb{Z}_+$ a demand function. There is a strongly polynomial time algorithm that finds (if there exist) $m(\mathcal{R})$ disjoint branchings so that $m(R_i)$ of them are spanning U_i with root-set R_i and each edge $e \in A$ is contained in at most $g(e)$ branchings.*

We also give a polyhedral description of arborescence packable subgraphs based on a connection with bi-set families and prove that the corresponding system of inequalities is totally dual integral (TDI).

Theorem 10 (B. and Frank [3]). *The linear system written for $x \in \mathbb{R}^A$*

$$\{0 \le x \le g,\ \varrho_x(Z) \ge p_1(Z) \text{ for every non-empty } Z \subseteq T\} \tag{2}$$

is TDI. In particular, the convex hull of arborescence-packable subgraphs of D is equal to the following polyhedron:

$$\{x \in \mathbb{R}^A :\ 0 \le x \le 1,\ \varrho_x(Z) \ge p_1(Z) \text{ for every non-empty } Z \subseteq T\}. \tag{3}$$

C_4-free 2-matchings

Let $G = (V, E)$ be an undirected graph and let $b : V \to \mathbb{Z}_+$ be an upper bound on the nodes. An edge set $F \subseteq E$ is called a **b-matching** if $d_F(v)$ is at most $b(v)$ for each node v. For some integer $t \ge 2$, by a **t-matching** we mean a b-matching with $b(v) = t$ for every $v \in V$. A closely related concept is **b-factor**, where instead of $d_F(v) \le b(v)$ strictly $d_F(v) = b(v)$ is required.

The so-called C_k-free and $C_{\le k}$-free 2-matching problems are important special cases of restricted b-matchings. A 2-matching M is **C_k-free** if it contains no cycle of length k, and it is **$C_{\le k}$-free**-free if it contains no cycle of length k or less. Papadimitriou showed that the problem of finding such subgraphs with maximum cardinality is NP-hard when $k \ge 5$ [12], and Hartvigsen [15] gave an augmenting path algorithm for the case $k = 3$. The C_4-free and $C_{\le 4}$-free 2-matching problems are left open.

The motivation of these problems is twofold: they have been studied as relaxations of the Hamiltonian cycle problem, while they are also strongly related to undirected node-connectivity augmentation. An interesting special case consists of increasing the connectivity by one, that is, when one would like to make a $(k-1)$-connected graph k-connected by adding a minimum number of new edges. We call this problem the **k-connectivity augmentation problem**[1].

We were interested in values of k close to n. If $k = n - 1$, then the graph should be simply extended to a complete graph; if $k = n - 2$, then the problem is equivalent to finding a maximum matching in the complement of the graph. It can be verified that the $(n-3)$-connectivity augmentation problem can be reduced to the problem of finding a square-free 2-matching of maximum size in a subcubic graph.

We give a polynomial time algorithm for the square-free 2-matching problem in simple subcubic graphs. Let γ_1 denote the time to solve the b-factor problem when $b(v) \le 2$.

Theorem 11 (B. and Kobayashi [5]). *In subcubic graphs, the square-free 2-matching problem can be solved in $O(n^3 \gamma_1)$ time.*

This leads to a polynomial time algorithm for the $(n-3)$-connectivity augmentation problem.

[1] The problem was recently solved in [24].

Theorem 12 (B. and Kobayashi [5]). *The $(n-3)$-connectivity augmentation problem is solvable in $O(n^3\gamma_1)$ time.*

Our algorithm is based on the theorem that square-free 2-matchings in a simple subcubic graph have a matroid-like structure called a jump system.

We also discuss the weighted versions of the problems. Given a $(k-1)$-connected graph $G = (V, E)$ and a weight function $w : \bar{E} \to \mathbb{R}_+$, where $\bar{E}$ is the complement of E, the **weighted k-connectivity augmentation problem** is the problem of finding a set of edges of minimum total weight that should be added to the original graph to obtain a simple k-connected graph. This problem is known to be NP-hard for fixed $k \geq 2$.

The weighted $(n-3)$-connectivity augmentation problem can be reduced to the problem of finding a square-free 2-matching maximizing the total weight of its edges, which we call the **weighted square-free 2-matching problem**. Z. Király proved that the weighted square-free 2-matching problem in bipartite graphs is NP-hard even for $0-1$ weights. This problem is, however, polynomially solvable in bipartite graphs if the weight function is node-induced on every square. For a subgraph $H = (V(H), E(H))$ of G, we say that w is **node-induced on H** if there exists a function $\pi_H : V(H) \to \mathbb{R}$ such that $w(e) = \pi_H(u) + \pi_H(v)$ for every edge $e = uv \in E(H)$.

We show that the weighted square-free 2-matching problem in simple subcubic graphs can be solved in polynomial time if the weight function is node-induced on every square. Suppose that for a weighted graph (G, w) and for a vector $x \in \{0, 1, 2\}^V$, we can find in γ_2 time an edge set $F \subseteq E$ maximizing $w(F)$ such that $d_F = x$.

Theorem 13 (B. and Kobayashi [5]). *In a weighted subcubic graph (G, w), if w is node-induced on every square in G, then the weighted square-free 2-matching problem is solvable in $O(n^3\gamma_2)$ time.*

We also show that the problem is NP-hard for general weights.

Theorem 14 (B. and Kobayashi [5]). *The weighted square-free 2-matching problem is NP-hard even if the given graph is cubic, bipartite, and planar.*

In our algorithm for the weighted problem, we use the theory of M-concave (M-convex) functions on constant-parity jump systems introduced by Murota.

$K_{t,t}$- and K_{t+1}-free t-matchings

The C_4-free 2-matching problem admits two natural generalizations. The first one is $K_{t,t}$-free t-matchings, while the second is t-matchings containing no complete bipartite graph $K_{a,b}$ with $a+b = t+2$. The latter variant corresponds to the problem of increasing the node-connectivity of a graph to $n-t-1$. In the dissertation, we consider the former variant of generalizations.

Let $\mathcal{K}$ be a set consisting of $K_{t,t}$'s, complete bipartite subgraphs of G on two colour classes of size t, and K_{t+1}'s, complete subgraphs of G on $t+1$ nodes. We give a min-max formula on the size of $\mathcal{K}$-free b-matchings and a polynomial time algorithm for finding one with maximum size under the assumptions that for any $K \in \mathcal{K}$ and any node v of K,

$$V_K \text{ spans no parallel edges} \tag{4}$$

$$b(v) = t \tag{5}$$

$$d_G(v) \le t + 1. \tag{6}$$

Note that this is a generalization of the maximum C_3-free, C_4-free and $C_{\le 4}$-free 2-matching problems in subcubic graphs.

We call a complete subgraph on four nodes **square-full** if it contains three forbidden squares. Note that, by assumption (6), every square-full subgraph is a connected component of G. We denote the number of square-full components of G by $S(G)$ for $t = 2$, and define $S(G) = 0$ for $t > 2$.

Theorem 15 (B. and Végh [7])**.** *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$ and $\mathcal{K}$ be a list of forbidden $K_{t,t}$ and K_{t+1} subgraphs of G so that (4), (5) and (6) hold. Then the maximum size of a $\mathcal{K}$-free b-matching is equal to the minimum value of*

$$b(U) + |E[W]| - |\dot{\mathcal{K}}[W]| + \sum_{T \in \mathcal{P}} \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|) \right\rfloor - S(G) \tag{7}$$

where U and W are disjoint subsets of V, $\mathcal{P}$ is a partition of the connected components of $G - U - W$ and $\dot{\mathcal{K}} \subseteq \mathcal{K}$ is a collection of node-disjoint forbidden subgraphs.

Among our assumptions, (4) and (5) may be considered as natural ones as they hold for the maximum $K_{t,t}$-free t-matching problem in a simple graph. However, the degree bound (6) is a restrictive assumption and dissipates essential difficulties. Our proof strongly relies on this and the theorem cannot be straightforwardly generalized.

The triangle-free 2-matching polytope

A cornerstone of matching theory is Edmonds' [13] description of the perfect matching polytope, the convex hull of incidence vectors of perfect matchings of a graph $G = (V, E)$. In the same paper, Edmonds gave the following characterization of the b-factor polytope.

We call $K \subseteq V, F \subseteq \delta(K)$ a **pair** if F does not contain loops (by notation, this only means restriction in case of $|K| = 1$). The pair is **odd** if $b(K) + |F|$ is odd. The **b-factor polytope** is the convex hull of the incidence vectors of b-factors of G.

Theorem 16 (Edmonds)**.** *The b-factor polytope is determined by*

$$\begin{aligned} &(i)\ 0 \le x_e \le 1 && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) = b(v) && (v \in V), \\ &(iii)\ x(\delta(K) \setminus F) - x(F) \ge 1 - |F| && ((K, F) \text{ odd}). \end{aligned} \tag{P_2}$$

A polyhedral description of b-matchings can easily be derived from Theorem 16.

Theorem 17. *The b-matching polytope is determined by*

$$\begin{aligned} &(i)\ 0 \le x_e \le 1 && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) \le b(v) && (v \in V), \\ &(iii)\ x(E[K]) + x(F) \le \lfloor \tfrac{b(K)+|F|}{2} \rfloor && ((K, F) \text{ odd}). \end{aligned} \tag{P_3}$$

Considering the maximum weight version of the C_k-free 2-factor problem, there is a firm difference between triangle- and square-free 2-factors. Z. Király showed [19] that finding a maximum weight square-free 2-factor is NP-hard even in bipartite graphs with $0-1$ weights. For subcubic graphs, polynomial time algorithms were given by Hartvigsen and Li [16], and by Kobayashi [20]. The former result implies that we should not expect a nice polyhedral description of the square-free 2-factor polytope. However, solvability of the triangle-free case was a main motivation of our investigation.

Deciding the existence of a triangle-free 2-factor becomes significantly harder without assuming the graph is subcubic. Let $\mathcal{T}$ be a list of forbidden triangles. Hartvigsen and Li gave a polyhedral description of the $\mathcal{T}$-free 2-factor polytope for subcubic simple graphs [16].

Theorem 18 (Hartvigsen and Li). *The $\mathcal{T}$-free 2-factor polytope of a simple subcubic graph is determined by*

$$\begin{aligned}
&(i)\ 0 \le x_e \le 1 && (e \in E),\\
&(ii)\ x(\delta(v)) = 2 && (v \in V),\\
&(iii)\ x(\delta(K)\setminus F) - x(F) \ge 1 - |F| && (K \subseteq V, F \subseteq \delta(K), |F| \text{ odd}),\\
&(iv)\ x(E_T) = 2 && (T \in \mathcal{T}).
\end{aligned} \qquad (P_5)$$

In the same paper, they gave a description of the $\mathcal{T}$-free 2-matching polytope as well and gave a sketch of the proof, which was published in its full version in [17].

As we have seen, the b-matching and b-factor polytopes have a similar description. Unexpectedly, the same does not hold in the triangle-free case. We say that a triangle T **1-fits** (resp. **2-fits**) a set $K \subseteq V$ if $|V_T \cap K| = 1$ (resp. 2). The **special edge** of a triangle T 1-fitting (resp. 2-fitting) the set K is the edge $e \in E_T$ having exactly 0 (resp. 2) endnodes in K, and is denoted by e_T. Given a set $\mathcal{T}$ of forbidden triangles, the set of triangles 1-fitting (resp. 2-fitting) K is denoted by $\mathcal{T}_K^1$ (resp. $\mathcal{T}_K^2$) while $\mathcal{T}_K$ stands for $\mathcal{T}_K^1 \cup \mathcal{T}_K^2$.

Definition 19. $(K, F, \mathfrak{T})$ is called a **tri-comb of Type i** if

1. $K \subseteq V$, $F \subseteq \delta(K)$, $\mathfrak{T} \subseteq \mathcal{T}_K^i$.
2. $F \cap E_{\mathfrak{T}} = \emptyset$.
3. The triangles in $\mathfrak{T}$ are edge-disjoint.

A tri-comb is called **odd** if $|F| + |\mathfrak{T}|$ is odd.

The fundamental result of Hartvigsen and Li is the following (see [16, 17]).

Theorem 20 (Hartvigsen and Li). *The $\mathcal{T}$-free 2-matching polytope of a simple subcubic graph is determined by*

$$\begin{aligned}
&(i)\ 0 \le x_e \le 1 && (e \in E),\\
&(ii)\ x(\delta(v)) \le 2 && (v \in V),\\
&(iii)\ x(E[K]) + x(F) + \textstyle\sum_{T\in\mathfrak{T}} x(E_T) \le |K| + \lfloor \frac{|F|+3|\mathfrak{T}|}{2} \rfloor && ((K, F, \mathfrak{T}) \text{ odd tri-comb of Type 2}),\\
&(iv)\ x(E_T) \le 2 && (T \in \mathcal{T}).
\end{aligned} \qquad (P_6)$$

Their proof is algorithmic and consists of clever triangle alteration and alternating forest growing. In [1], we give new proofs of Theorems 18 and 20 in a slightly more general form.

Splitting property via shadow systems

Let $\mathcal{P} = (P, \prec)$ be a finite partially ordered set. For a subset $H \subseteq P$, sets $\mathcal{U}(H) = \{x \in P : \exists h \in H : x \succeq h\}$ and $\mathcal{L}(H) = \{x \in P : \exists h \in H : x \preceq h\}$ are called the upper and lower shadows of H, respectively. We say that a maximal antichain A has the **splitting property** if it can be partitioned into two disjoint parts $A_1 \cup A_2 = A$ such that $\mathcal{U}(A_1) \cup \mathcal{L}(A_2) = P$. A maximal antichain $A \subseteq P$ is called **dense** if it satisfies the following: whenever $x \prec a \prec y$ for some $a \in A$ and $x, y \in P$, there exists an $a' \in A \setminus \{a\}$ also satisfying $x \prec a' \prec y$. Ahlswede et al. proved the following theorem [8].

Theorem 21 (Ahlswede, Erdős and Graham)**.** *Every dense maximal antichain in a finite poset satisfies the splitting property.*

We consider the poset of multisets of k colours. Formally, let us use the elements of the group $\mathbb{Z}_k$ as colours, denoted by $\{1, \dots, k\}$. We call the vectors $\mathbb{Z}_k \to \mathbb{Z}$ k**-colour vectors**, and denote their set by M_k. We can define a natural partial ordering on M_k: for $a, c \in M_k$, $a \prec c$ if $a_i \leq c_i$ for every $i \in \mathbb{Z}_k$ and $a \neq c$. If $a \prec c$, we also say that a is a **shadow** of c. Let

$$M_k^r = \{x \in M_k, \sum_{i \in \mathbb{Z}_k} x_i = r\}$$

denote the set of k-colour vectors whose coordinates sum up to r.

One of our main results shows the splitting property of this antichain for $r = k$.

Theorem 22 (B., Csikvári, Kovács and Végh [2])**.** *In the poset $(M_k, \prec)$, the maximal antichain M_k^k has the splitting property, that is, M_k^k can be partitioned into disjoint sets A_1 and A_2 such that $\mathcal{U}(A_1) \cup \mathcal{L}(A_2) = M_k$.*

It is easy to verify that M_k^k is not dense and therefore Theorem 21 does not imply our result.

For $r \leq t \leq n$, a **Turán** (n, t, r)**-system** is an r-uniform hypergraph on n nodes such that every t-element subset of the nodes spans at least one edge of the hypergraph. The Turán number $T(n, t, r)$ asks for the minimum size of such a family; determining the exact values is a problem posed by Pál Turán [22].

The limit

$$t(t, r) = \lim_{n \to \infty} \frac{T(n, t, r)}{\binom{n}{r}}$$

expresses the fraction of all r-element subsets needed for a Turán (n, t, r)-system. No exact value is known for any $t > r > 2$ - in 1981, Pál Erdős offered a bounty of \$500 for even a single special case and \$1000 for resolving the general case. The best currently known upper bound is due to Sidorenko [21].

Theorem 23 (Sidorenko)**.** *For any integers $t > r$,*

$$t(t, r) \leq \left(\tfrac{r-1}{t-1}\right)^{r-1}. \tag{8}$$

In Theorem 22, the required property of $A_1 \subset M_k^k$ is that for every vector $c \in M_k^{k+1}$, A_1 must contain at least one shadow of A_1. Generalizing this notion, for $r < t$ we call $A \subseteq M_k^r$ a $(t, r; k)$**-shadow system**, if for every colour vector $c \in M_k^t$, A contains at least one shadow of c. With this terminology, A_1 in Theorem 22 is a $(k+1, k; k)$-shadow system. Consider a vector $s \in \mathbb{Z}_k^r$. The **colour profile** $a = M(s) \in M_k^r$ can be naturally defined so that a_i equals the number of i's in s for $1 \leq i \leq k$.

We give a new interpretation of Sidorenko's construction in terms of shadow systems, and reprove the theorem using the following combinatorial colouring result.

Theorem 24 (B., Csikvári, Kovács and Végh [2]). *For integers $t > r$, there exists a $(t, r; t-1)$-shadow system $\mathcal{A}_r^t \subseteq M_{t-1}^r$ so that if we pick a vector $s \in \mathbb{Z}_{t-1}^r$ uniformly at random, then the probability of $M(s) \in \mathcal{A}_r^t$ equals $\left(\frac{r-1}{t-1}\right)^{r-1}$.*

We also introduced the natural weighted extension of Turán numbers: we are given a nonnegative weight function w on the r-element subsets of V, and let w^* denote the total weight of all subsets. The **Turán weight** $T_w(n, t, r)$ is the minimum weight of a Turán (n, t, r)-system. Analogously to $t(t, r)$ we may define

$$tw(t, r) = \lim_{n\to\infty} \sup_w \frac{T_w(n, t, r)}{w^*}.$$

Somewhat surprisingly, we show that $tw(t, r) = t(t, r)$, that is, the bound is not affected by the weight, and the bound on $tw(t, r)$ can be derived from Theorem 24 the same way as the bound on $t(t, r)$.

Theorem 25 (B., Csikvári, Kovács and Végh [2]). *For any integers $t > r$, we have $tw(t, r) = t(t, r)$.*

There is a strong connection between Turán systems and Tuza's [23] famous conjecture asserting that in every graph the minimum number of edges covering every triangle is at most twice the maximum number of pairwise edge-disjoint triangles. Finding a minimum number of edges in a graph $G = (V, E)$ covering every triangle is equivalent to computing the weighted Turán number $T_w(n, 3, 2)$ with $n = |V|$, and $w(e) = 1$ if $e \in E$ and $w(e) = 0$ otherwise.

Now let $H = (V, \mathcal{E})$ be a simple $(r-1)$-uniform hypergraph and $w : \mathcal{E} \to \mathbb{R}_+$ a weight function. A complete subhypergraph on r nodes is called an r**-block**. The minimum weight of a set of hyperedges covering each r-block is denoted by $\tau_w(H)$, while the maximum number of r-blocks such that each hyperedge e is contained in at most $w(e)$ of them is denoted by $\nu_w(H)$.

We propose the following weighted hypergraphic version of Tuza's conjecture.

Conjecture 26 (B., Csikvári, Kovács and Végh [2]). *Let $H = (V, \mathcal{E})$ be a simple $(r-1)$-uniform hypergraph and $w : \mathcal{E} \to \mathbb{R}_+$ a weight function. Then $\tau_w(H) \le \lceil \frac{r+1}{2} \rceil \nu_w(H)$.*

Although Conjecture 26 is still open, we manage to prove its fractional relaxation.

Theorem 27 (B., Csikvári, Kovács and Végh [2]). *Let $H = (V, \mathcal{E})$ be a simple $(r-1)$-uniform hypergraph and $w : \mathcal{E} \to \mathbb{R}_+$ a weight function. Then $\tau_w(H) \le (r-1)\tau_w^*(H)$.*

Theorem 27 extends the result of Krivelevich on the fractional version of Tuza's original conjecture and also makes use of our construction on shadow systems.

The thesis is based on the following papers

[1] K. Bérczi. The triangle-free 2-matching polytope of subcubic graphs. Technical Report TR-2012-02, Egerváry Research Group, Budapest, 2012. `www.cs.elte.hu/egres`. 7

[2] K. Bérczi, P. Csikvári, E. R. Kovács, and L. A. Végh. Splitting property via shadow systems. Technical Report TR-2013-02, Egerváry Research Group, Budapest, 2013. `www.cs.elte.hu/egres`. 8, 9

[3] K. Bérczi and A. Frank. Packing arborescences. *Lecture Notes, in: RIMS Kokyuroku Bessatsu B*, 23:1–31, 2009. 2, 4

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